Semester-VI

Paper Code: CMA-316

Paper Name: Ring Theory & Linear Algebra

Full Mark: 80

1. Write the correct answer:(1×3)

(i) In the ring R[x], if f(x), g(x)∈R[x] and deg(f(x)) = m, deg(g(x)) = n. Then deg(f(x).g(x)) is

(A) less than or equal to max(m,n)

(B) less than or equal to m+n

(C) less than or equal to m·n

(D) less than or equal to max(m+n,m·n).

(ii) Let W be a subset of a linear space V(F). Then the annihilator A(W) is

(A) a subspace of W(F)

(B) a subspace of V(F)

(C) a subspace of $\widehat{V}$(F), 1st dual

(D) a subspace of $\widehat{V}$(F), 2nd dual.

(iii) In an inner product space V($\mathbb{R}$), if (x,z) = (y,z) ∀ x,y,z ∈ V($\mathbb{R}$). Then

(A) x=y

(B) y=z

(C) z=x

(D) z = x-y.

2. Answer all the questions:(1×6)

(i) Define a Principal Integral Domain (PID).

(ii) When is a polynomial f(x)∈R[x] said to be irreducible?

(iii) Define an eigen space associated with a linear operator T and with an eigen value C of T.

(iv) What is meant by the statement that a subset W is T - invariant subspace of a vector space V(F).

(v) Define a Unique Factorisation Domain (UFD).

(vi) Write the triangle inequality in an inner product space.

3. Answer any five (5) of the following:(3×5)

(i) Prove that an arbitrary ring R can be imbedded into the ring R[x].

(ii) Let T:$C^2 \to C^2$ be defined by T(x,y) = (x,0).

If α = {α_1 = (1,0), α_2 = (0,1)}

β = {β_1 = (1,i), β_2 = (-i,2)}

Compute $[T]_{\alpha,\beta}$.

(iii) Let T be a linear operator defined on a FDVS V(F). Let $C_1, C_2, ...C_n$ and $V_1, V_2, ...V_n$ are distinct eigen values of T and the corresponding eigen vectors. Prove that $V_1, V_2, ...V_n$ are linearly independent.

(iv) In an inner product space V(F), if for (x, y) ∈F, $x \perp y$ then show that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

(v) R[x] is commutative implies R is commutative and conversely.

(vi) Using Cauchy Schwarz's inequality, prove that cosine of an angle is of absolute value atmost 1.

4. Answer any five (5) of the following:(4×5)

(i) Prove that an Euclidean Domain is obviously a PID.

(ii) In a UFD, R an element is prime if and only if it is irreducible.

(iii) Let $\mathbb{R}^2$ = {(x,y) : x∈ $\mathbb{R}$, y∈ $\mathbb{R}$}. Define T : $\mathbb{R}^2 \to \mathbb{R}^2$ by T(x,y) = (x,0).
Show that 1 is an eigen value of T and the eigen space of 1 is the x-axis.

(iv) Let T be a linear operator on a FDVS V(F). Prove that a scalar $\alpha \in$ F is an eigen value of T if and only if T-αI is singular (not invertible).

(v) Let $\{u_1, u_2, ...u_n\}$ be an orthonormal set in V(F). Show that for any $v \in$V(F),
$w = v$ -$\sum_{i=1}^{n}(v, u_i)u_i$ is orthogonal to each of u_i, i = 1,2,3,.....n.

(vi) Obtain an orthonormal basis w.r.t. standard inner product for the subspace of $\mathbb{R}^3$ generated by (1,0,3) and (2,1,1).

5. Answer any two (2) of the following:(6×2)

(i) State and prove Gauss lemma.

(ii) Show that $8x^3-6x-1$ and $x^4+x^3+x^2+x+1$ are irreducible by Eisenstein's criterion on irreducibility.

(iii) For any prime p, show that the polyniomial $f(x) = x^{p-1} + x^{p-2} +$ .... $+ x^2 + x + 1$ is irreducible over $\mathbb{Q}$, the set of rational numbers.

6. Answer any two (2) of the following:(6×2)

(i) Prove that the characteristic polynomial and the minimal polynomial of the matrix A given by

$$A = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix} \text{ if } a, b, c \; are \; scalars$$

are equal.

(ii) Show that the matrix

$$A = \begin{pmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ 2 & -1 & 1 \end{pmatrix} \text{ is diagonalisable.}$$

(iii) Obtain eigen values, eigen vectors and eigen space of

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

7. Answer any two (2) of the following:(6×2)

(i) State and prove the Bessel's inequality.

(ii) State and prove the Gram - Schmidt Orthogonalization process.

(iii) If V(F) be a FDVS and W is a subspace, then prove that V(F) = W $\oplus$ W$^{\perp}$.